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A STUDY OF THE STABILITY OF REINFORCED CYLINDRICAL AND CONICAL SHELLS SUBJECTED TO VARIOUS TYPES AND COMBINATIONS OF LOADS

SECTION II - Stress in a Segment of A Conical Shell Subjected to Lateral Normal Load

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ABSTRACT

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Following the exact linear theory of conical shells given in Flugge's "Stresses in Shells", a solution for a segment of an isotropic truncated conical shell with linearly varying thickness subjected to an arbitrary lateral normal load is obtained. The segment is clamped at the smaller circular end and free at the other end. The straight edges lying in two meridians are assumed to be free from bending and normal force. AUTHOR

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STRESSES IN A SEGMENT OF A CONICAL SHELL SUBJECTED TO LATERAL NORMAL LOAD

by

Chin Hao Chang, Ph.D.*

I. INTRODUCTION

The study of a segment of a conical shell is of interest to the sponsoring agency as the study relates to the stress analysis of engine shrounds. A shroud is a segment of a conical shell subjected to lateral normal load and temperature change. The smaller circular edge may be considered clamped and the other three edges free.

The analytical investigation of such a structure may be divided into two phases. The first phase is concerned with the lateral normal load; the second, with the temperature change. The present report is the result of a study of the first phase only. The other phase requires further investigation.

A systematic literature search for the formulations and solutions of problems concerned with conical shells subjected to

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normal loads has been made. Flugge in his recent book, Reference (1) "Stresses in Shells," presents the most complete and directly applicable basic formulations of the problem. This formulation, which is exact within the frame-work of linear theory of elasticity, will be followed in the present work.

The available solutions of homogenous conical shell problems can be classified into two categories. The first category is for shells with constant thickness. The second is for those having a thickness directly proportional to the distance measured from the apex along the surface. This distance will be denoted by s. All the solutions are functions of two variables; one variable is s, the other variable is the angle θ measured in a plane perpendicular to the geometric axis of the cone. In all of the solutions, the variables are separable, so that the solutions are expressed in terms of Fourier series in θ with functions of the other variables as coefficients. In the first category, if there is axial symmetry, seen from Reference (2) and (3), the coefficients are Bessels functions while if there is axial asymmetry seen from References (4) to (7), the coefficient are power series. As it is pointed out in the discussion of Reference (5) and also in (6) these power-series converge so slowly that the solutions obtained are of no practical use. In the second category, seen from Reference (1), regardless of whether or not axial symmetry exists, the coefficients are given in closed forms.

In the present problem, there is axial asymmetry. The thickness of shell could be either constant or linearly varying. The shell segment is quite thin and is far away from the apex. Hence, if the shell thickness varies the rate of change of the thickness is very small. For simplicity the thickness will be assumed to vary linearly from the apex. The shell is also assumed to be isotropic. Nevertheless, if, it is of interest to the sponsoring agency, as the next step, a segment of a conical shell with constant thickness and of orthotropic material will be considered.

The conical panel studied here, as mentioned before is a cantilevered one subjected to lateral normal load with the smaller circular-arc end clamped and the other end free. The two straight edges lying on two meridians are first considered as free edges. However, considering these edges free causes some trouble in seeking an analytical solution. As has been pointed out in the case of a panel cut from a cylindrical shell in Reference (1) p. 239: "it has so far not been possible, and probably never will be with simple mathematical means, to find a solution which can satisfy any ~~desired set of boundary conditions along all four edges of a~~ rectangular panel cut from a cylincrical shell." It seems that this is also true in the present case. After an analytical examination and a consideration of practical convenience, the straight edges are modified by attaching a bar to each of them. The bar will carry the axial and transverse shearing force but leave the shell edges free to rotate and to move in the tangential directions. In what follows, a solution will be obtained which will satisfy these modified boundary conditions along each of the straight edges.

The shell segment studied is of isotropic and homogeneous material. However, the result could be used for a rational design of a stiffened shell provided the stiffeners are closely spaced in both directions. A detailed procedure for applying the result to a stiffened shell is given in Appendix I.

II. SOLUTION

Consider a segment of a truncated thin conical shell of elastic isotropic and homogeneous material, whose middle surface is described by the co-ordinates s and θ . Let s be the distance measured from the apex along the conical surface as shown in Fig.1, and θ be the angle measured from a fixed meridian to another one. The inclination of s with respect to the geometrical axis is indicated by the angle α shown in the figure. The end of $s=L_1$ is fixed and that of s=L is free. The edges lying along the two meridians of $\theta =0$ and $\theta = \theta_1$ are first considered to be free. Let u, v, and w be three displacement components in s, θ and normal to the middle surface directions respectively. The elastic law assumes the following relationships between the forces and displacements:

$$N_s = D\left[v^{\bullet} + \frac{\nu}{s}(u' \sec\alpha + v + w\tan\alpha)\right] - K\frac{w^{\bullet\bullet}}{s}\tan\alpha$$

$$N_\theta = D\left[\frac{1}{s}\left(\frac{u'}{\cos\alpha} + v + w\tan\alpha\right) + \nu v^{\bullet}\right]$$

$$+ K\frac{1}{s^3}\left[v\tan\alpha + w\tan^2\alpha + w''\sec^2\alpha + s w^{\bullet}\right]\tan\alpha$$

$$N_{s\theta} = D\,\frac{1-\nu}{2}\left[u^{\bullet} - \frac{u}{s} + \frac{v'}{s\cos\alpha}\right]$$

$$+ K\,\frac{1-\nu}{2}\,\frac{1}{s^3}\left[s u^{\bullet} - u - s w'^{\bullet}\frac{1}{\sin\alpha} + w'\frac{1}{\sin\alpha}\right]\tan^2\alpha$$

$$N_{\theta s} = D\,\frac{1-\nu}{2}\,\frac{1}{s}\left[s u^{\bullet} - u + v'\sec\alpha\right]$$

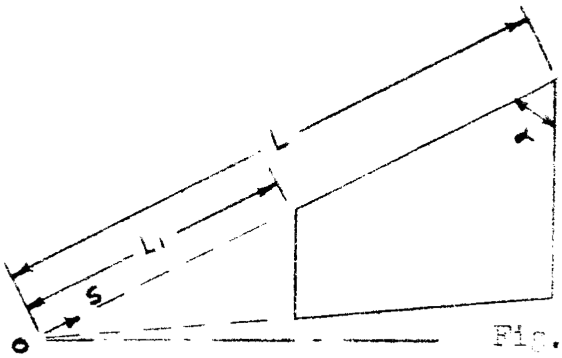
$$+ K\,\frac{1-\nu}{2}\,\frac{1}{s^3}\left[v'\sec\alpha + s w'^{\bullet}\,\frac{1}{\sin\alpha} - w'\frac{1}{\sin\alpha}\right]\tan^2\alpha$$

$$M_s = K\frac{1}{s^2}\left[s^2 w^{\bullet\bullet} - s v^{\bullet}\tan\alpha + \nu(w''\sec^2\alpha + s w^{\bullet} - u'\sec\alpha\tan\alpha)\right]$$

$$M_\theta = K\frac{1}{s^2}\left[W''\sec^2\alpha + sW^{\cdot} + W\tan^2\alpha + V\tan\alpha + \nu s^2 W^{\cdot\cdot}\right]$$

$$M_{s\theta} = K(1-\nu)\frac{1}{s^2}\left[sW'^{\cdot} - W' - su'\sin\alpha + u\sin\alpha\right]\sec\alpha$$

$$M_{\theta s} = K(1-\nu)\frac{1}{s^2}\left[sW'^{\cdot} - W' - \frac{1}{2}su'\sin\alpha + \frac{1}{2}u\sin\alpha + \frac{1}{2}V'\tan\alpha\right]\sec\alpha \qquad (1a\text{-}h)$$



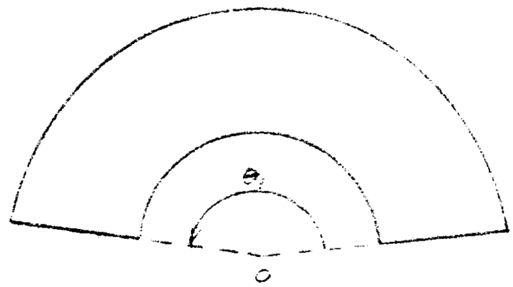
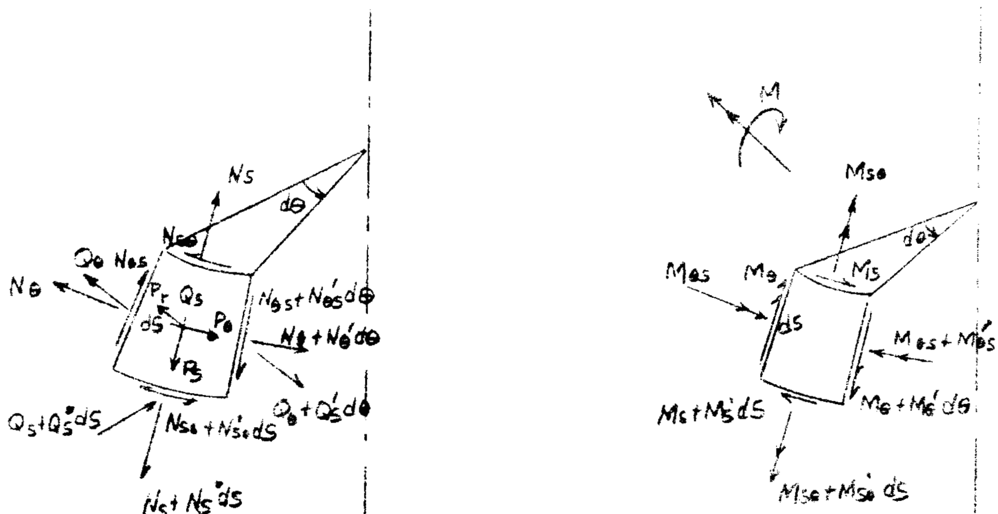


Fig.1



(2a)

(2b)

Fig.2

where $N_s, \ldots\ldots, M_{\theta s}$ are forces and moments per unit length acting on the sections of elements as shown in Fig. 2. All forces and moments shown in the figures are positive. The dot symbol indicates the partial differentation with respect to s, and prime, differentiation with respect to θ . For instance,

$$u' = \frac{\partial u}{\partial S} \qquad u'^{\bullet} = \frac{\partial^2 u}{\partial \theta \partial s}$$

The right-hand rule is applied to the double-arrow head which represents moment in Fig. 2b. D and K are constants of rigidity defined by

$$D = \frac{Et}{1-\nu^2} \quad \text{and} \quad K = \frac{Et^3}{12(1-\nu^2)} \tag{2}$$

where E is Young's modulus of elasticity, ν is Poisson's ratio, and t, the thickness of the shell.

Applying the condition of equilibrium, the following six equations are obtained:

$$\begin{aligned}
&(SN_s)^{\bullet} + N'_{\theta s} \sec\alpha - N_\theta = -P_s S \\
&S(N_{s\theta})^{\bullet} + N'_\theta \sec\alpha + N_{\theta s} - Q_\theta \tan\alpha = -P_\theta S \\
&N_\theta \tan\alpha + Q'_\theta \sec\alpha + (SQ_s)^{\bullet} = P_r S \\
&(SM_s)^{\bullet} + M'_{\theta s} \sec\alpha - M_\theta = SQ_s \\
&(SM_{s\theta})^{\bullet} + M'_\theta \sec\alpha + M_{\theta s} = SQ_\theta \\
&S(N_{\theta s} - N_{s\theta}) = M_{\theta s} \tan\alpha
\end{aligned} \tag{3a-f}$$

where P_s and P_θ are tangential surface loads in s and θ directions respectively, P_r , the normal surface load acting on an element per unit of area.

The last equation in (3) is an identity which can be shown through the elastic law (1). This equation, thus, can be dropped. Using equations (3d) and (3e) to eliminate the transverse shearing forces Q_s and Q_θ in the other three equations, one finally obtains three equations of equilibrium. They are

$$s(sN_{s\theta})^{\cdot} + sN'_{\theta}\sec\alpha + sN_{\theta s} - (sM_{s\theta})^{\cdot}\tan\alpha - M_{\theta s}\tan\alpha - M'_{\theta}\tan\alpha\sec\alpha = -p_\theta s^2 ,$$

$$(sN_s)^{\cdot} + N'_{\theta s}\sec\alpha - N_\theta = -p_s s ,$$

$$sN_\theta\tan\alpha + s(sM_s)^{\cdot\cdot} + (sM'_{s\theta})^{\cdot}\sec\alpha + (sM'_{\theta s})^{\cdot}\sec\alpha + M''_{\theta}\sec^2\alpha - sM^{\cdot}_{\theta} = p_r s^2 . \qquad (4a\text{-}c)$$

It has been mentioned before that for simplicity the thickness of the shell is assumed to be in direct proportion to the distance s but independent to θ .

$$t = \delta \cdot s \qquad (5)$$

Substituting this expression into (2), one observes

$$\frac{K}{D} = ks^2$$

where

$$k = \frac{\delta^2}{12} \qquad (6)$$

is a very small number because δ , defined in (5) usually is a small constant.

If the segment is subject to the normal load only, the equations of equilibrium may be expressed by means of the elastic law in terms of the three displacement components in the following form:

$$\frac{1-\nu}{2} s^2 u^{\cdot\cdot} + u'' \sec^2\alpha + (1-\nu) s u^{\cdot} - (1-\nu) u + \frac{1+\nu}{2} s v'^{\cdot} \sec\alpha + (2-\nu) v' \sec\alpha + w' \tan\alpha \sec\alpha + k\left[\frac{3}{2}(1-\nu) s^2 u^{\cdot\cdot} \tan\alpha + 3(1-\nu) s u^{\cdot} \tan\alpha - 3(1-\nu) u \tan\alpha - \left(\frac{3-\nu}{2}\right) s^2 w'^{\cdot\cdot} \sec\alpha - 3(1-\nu) s w'^{\cdot} \sec\alpha + 3(1-\nu) w' \sec\alpha\right] \tan\alpha = 0 \;;$$

$$\frac{(1+\nu)}{2} s u'^{\cdot} \sec\alpha - \frac{3}{2}(1-\nu) u' \sec\alpha + s^2 v^{\cdot\cdot} + \frac{1-\nu}{2} v'' \sec\alpha + 2 s v^{\cdot} - (1-\nu) v + \nu s w^{\cdot} \tan\alpha - (1-\nu) w \tan\alpha + k\left[\frac{1-\nu}{2} v'' \tan\alpha \sec^2\alpha - v \tan\alpha - s^3 w^{\cdot\cdot\cdot} + \frac{1-\nu}{2} s w''^{\cdot} \sec^2\alpha - 3 s^2 w^{\cdot\cdot} - \frac{3-\nu}{2} w'' \sec^2\alpha - s w^{\cdot} - w \tan^2\alpha\right] \tan\alpha = 0 \;; \qquad \text{(7a-c)}$$

$$\left[u' \sec\alpha + \nu s v^{\cdot} + v + w \tan\alpha\right] \tan\alpha + k\left[-\frac{3-\nu}{2} s^2 u'^{\cdot\cdot} \sec\alpha - (3+\nu) s u'^{\cdot} \sec\alpha + (3-5\nu) u' \sec\alpha - s^3 v^{\cdot\cdot\cdot} + \frac{1-\nu}{2} s v''^{\cdot} \sec^2\alpha - 6 s^2 v^{\cdot\cdot} + (2-\nu) v'' \sec^2\alpha - 7 s v^{\cdot} - v(1-\tan^2\alpha)\right] \tan\alpha + k\left[s^4 w^{\cdot\cdot\cdot\cdot} + 2 s^2 w''^{\cdot\cdot} \sec^2\alpha + w'''' \sec^4\alpha + 8 s^3 w^{\cdot\cdot\cdot} + 4 s w''^{\cdot} \sec^2\alpha + s^2(11+3\nu) w^{\cdot\cdot} + 2 w'' \tan^2\alpha \sec^2\alpha - (5-6\nu) w'' \sec^2\alpha - 2(1-3\nu) s w^{\cdot} - w(1-\tan^2\alpha)\right] \tan^2\alpha = P_r \frac{s^2}{D}$$

The above formulation of the problem is given in Reference (1).

The segment has the following boundary conditions: Along $s=L_1$ it is clamped so that all the displacement components and also the rate of change of w with respect to s must vanish. i.e.

$$u = v = w = 0, \qquad \frac{\partial w}{\partial s} = 0 \qquad \text{for } s = L_1 \qquad (8)$$

Along the edge s = L, the segment is free from all stresses so that the normal moment (M_s), normal force (N_s), transverse and tangential shearing forces (Q_s and $N_{s\theta}$) and twist moment ($M_{s\theta}$)

must vanish. The vanishing of the first two forces is quite obvious. Simply

$$M_s = 0 \text{ and } N_s = 0 \qquad \text{for } s = L \tag{9}$$

The other three forces, according to Kirchhoff's law in the theory of plates may be combined into two by considering the twisting moment to be composed of shearing forces. Introducing S_s and T_s as the resultant shearing forces in transverse and tangential directions respectively, one has*

$$\begin{aligned} S_s &= Q_s + \frac{1}{s}\frac{\partial M_{s\theta}}{\partial \theta}\sec\alpha \\ T_s &= N_{s\theta} - \frac{M_{s\theta}}{s}\tan\alpha \end{aligned} \tag{10}$$

Hence, one requires

$$S_s = 0 \text{ and } T_s = 0 \qquad \text{for } s = L \tag{11}$$

The boundary conditions along the two straight edges, as mentioned before, are also considered to be free. Then along these edges,

$$\left.\begin{aligned} M_\theta &= 0 \\ N_\theta &= 0 \\ T_\theta &= N_{\theta s} = 0 \\ S_\theta &= Q_\theta + \frac{\partial M_{\theta s}}{\partial s} = 0 \end{aligned}\right\} \qquad \text{for } \theta = 0 \text{ and } \theta_1. \tag{12a-d}$$

It is impossible to have all of the above conditions satisfied. In order to compare the significance of these four quantities, an examination of their orders of magnitude will be taken in what follows.

It is observed from Eq.(5) that δ is a very small constant because, in the present case, s is much larger than t. i.e. $\delta \ll 1$.

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* Refer to p.233 reference (1)

It should also be noted that u, v and w as well as the products of s^m with the mth derivatives of u, v and w with respect to s are of the same order as t, where m is an integer. In fact, this assumption is essential to the present theory. Thus it can be seen from equations (1b), (1d) and (1f) that, considering the leading terms,

$$N_\theta = 0\ (\delta)$$
$$N_{\theta s} = 0\ (\delta) \tag{13a-c}$$
$$M_\theta = 0\ (\delta^3 s) = 0\ (\delta^2)$$

and from equations (12d), (3d) and (1g),

$$S_\theta = 0\ (Q_s) = 0\ \left(\frac{M_{s\theta}}{s}\right) = 0\ (\delta^3) \tag{13-d}$$

where the notation 0 (δ) means that the function is of the order of δ, etc. Thus the transverse shearing force, S_θ, is of the highest order of δ. Therefore, it is negligible or it may be provided by a comparatively weak support. Further, among the other three in (13), the tangential shearing force $N_{\theta s}$ is easily provided by attaching a relatively inextensible bar to each of the straight edges. By requiring this bar to have a certain degree of stiffness in the normal to the middle surface direction, it may provide the resistance to both S_θ' and $N_{\theta s}$. This modification allows the edges to be only partially free. Hence, along these partially free edges, one requires that

$$N_\theta = 0 \text{ and } M_\theta = 0 \quad \text{for } \theta = 0 \text{ and } \theta_1 . \tag{14}$$

Observation of equations (1b) and (1f) reveals that the above two conditions are satisfied by assuming

$$u = \sum_{n=1}^{\infty} A_n f_n(s) \cos\frac{n\pi\theta}{\theta_1},\quad v = \sum_{n=1}^{\infty} B_n f_n(s) \sin\frac{n\pi\theta}{\theta_1},\quad w = \sum_{n=1}^{\infty} C_n f_n(s) \sin\frac{n\pi\theta}{\theta_1} \tag{15}$$

where A_n, B_n, and C_n are constants, and $f_n(s)$ is a function of s to be determined by the set of differential equations (7). (Note when n=0.)

(15) represents a configuration of twisting due to a distributed tangential shearing force applied at the free end at s=L, which is of no interest in the present case.)

The normal surface load P_r may also be expressed in Fourier series

$$P_r = P(s) \sum_{n=1}^{\infty} a_n \sin \frac{n\pi\theta}{\theta_1} \tag{16}$$

where P (s) is a given function, and a_n are Fourier coefficients computable from a given load distribution in θ -direction.

For later convenience, a nondimensional variably is introduced such that

$$y = \sqrt{\frac{s}{L}} \tag{17}$$

along with the assumption that

$$f_n(s) = y^{\lambda_n - 1} \tag{18}$$

in which λ_n is a constant to be determined. Substituting (15) into (7), one requires

$$\begin{aligned} &d_{11} A_n + d_{12} B_n + d_{13} C_n = 0 \\ &d_{21} A_n + d_{22} B_n + d_{23} C_n = 0 \\ &y^{\lambda_n - 1} [d_{31} A_n + d_{32} B_n + d_{33} C_n] = \frac{L^2}{D} a_n y^4 P(y) \quad , \end{aligned} \tag{19a-c}$$

where

$$\begin{aligned} d_{11} &= d_{11}^2 \lambda_n^2 + d_{11}^0 \\ d_{12} &= d_{12}' \lambda_n + d_{12}^0 \\ d_{13} &= d_{13}^2 \lambda_n^2 + d_{13}' \lambda_n + d_{13}^0 \\ d_{22} &= d_{22}^2 \lambda_n^2 + d_{22}^0 \end{aligned} \tag{20}$$

$$d_{23} = d_{23}^3 \lambda_n^3 + d_{23}^2 \lambda_n^2 + d_{23}^1 \lambda_n + d_{23}^0$$

$$d_{33} = d_{33}^4 \lambda_n^4 + d_{33}^2 \lambda_n^2 + d_{33}^0$$

while d_{21}, d_{32} and d_{31} are obtained by replacing λ_n, by $-\lambda_n$ in d_{12}, d_{23}, d_{13} respectively; where

$$\begin{aligned}
d_{11}^2 &= \frac{1-\nu}{8}[1 + 3k \tan^2\alpha] \\
d_{11}^0 &= -[(1-\nu)\tfrac{9}{8}(1+3k\tan^2\alpha) + n^2 \sec^2\alpha] \\
d_{12}^1 &= \frac{1+\nu}{4} n \sec\alpha \\
d_{12}^0 &= \tfrac{1}{4}(7-5\nu) n \sec\alpha \\
d_{13}^2 &= -\tfrac{k}{8}(3-\nu) n \tan\alpha \sec\alpha \\
d_{13}^1 &= k\nu n \tan\alpha \sec\alpha \\
d_{13}^0 &= [\tfrac{3}{8}k(9-11\nu)+1] n \tan\alpha \sec\alpha \\
d_{22}^2 &= \tfrac{1}{4} \\
d_{22}^0 &= -[\tfrac{1}{4} + (1-\nu)(1+\tfrac{1}{2}n^2\sec^2\alpha) + k\tan^2\alpha\,(1+\tfrac{1-\nu}{2}n^2\sec^2\alpha)] \\
d_{23}^3 &= -\tfrac{1}{8}k\tan\alpha \\
d_{23}^2 &= \tfrac{3}{8}k\tan\alpha \\
d_{23}^1 &= -\tfrac{1}{8}k\tan\alpha[3+2(1-\nu)n^2\sec^2\alpha] + \tfrac{\nu}{2}\tan\alpha \\
d_{23}^0 &= -\tfrac{1}{2}\tan\alpha(2-\nu) + \tfrac{1}{8}k\tan\alpha[1-8\tan^2\alpha + 2(7-3\nu)n^2\sec^2\alpha] \\
d_{33}^4 &= -\tfrac{k}{16} \\
d_{33}^2 &= \tfrac{k}{8}[7-6\nu+4n^2\sec^2\alpha]
\end{aligned} \tag{21}$$

$$d_{33}^{0} = -\Big\{ \tan^2\alpha + \frac{K}{16}\big[(13-12\nu) - 16(1-\tan^2\alpha)\tan^2\alpha$$

$$+ 8(11-12\nu-4\tan^2\alpha)n^2\sec^2\alpha + 16n^4\sec^4\alpha\big]\Big\}$$

All d_{ij}^{k} are constants depending on the parameters k, ν α and n.

The homogeneous solutions are obtained from a set of equations formed by setting the term on the right hand side of equations (19) equal to zero, yielding

$$\begin{aligned} d_{11}A_n + d_{12}B_n + d_{13}C_n &= 0 \\ d_{21}A_n + d_{22}B_n + d_{23}C_n &= 0 \\ d_{31}A_n + d_{32}B_n + d_{33}C_n &= 0 \end{aligned} \qquad (22)$$

The necessary and sufficient condition for the constants A_n, B_n and C_n to be non-vanishing is that the determinant formed by the coefficients of the three simultaneous equations vanish, i.e.

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = 0 \qquad (23)$$

Substituting the coefficients given by (20) into the determinant, one has

$$\lambda_n^8 + g_6\lambda_n^6 + g_4\lambda_n^4 + g_2\lambda_n^2 + g_0 = 0 \qquad , \qquad (24)$$

where

$$g_6 = \frac{1}{\Delta}\Big\{ d_{11}^{2}\big[d_{22}^{0}d_{33}^{4} + d_{22}^{2}d_{33}^{2} + 2d_{23}^{2}d_{23}^{1} - d_{23}^{2}d_{23}^{2}$$

$$+ d_{11}^{0}\big[d_{22}^{2}d_{33}^{4} + d_{23}^{3}d_{23}^{3}\big] - d_{13}^{2}d_{13}^{2}d_{22}^{2}$$

$$+ d_{12}^{2}\big[2d_{13}^{2}d_{23}^{3} + d_{12}^{1}d_{33}^{4}\big]\Big\}$$

$$g_4 = \frac{1}{\Delta}\Big\{ d_{11}^0\left[d_{22}^2 d_{33}^2 - 2d_{23}^3 d_{23}^1 + d_{23}^2 d_{23}^2 + d_{22}^0 d_{33}^4\right]$$
$$+ d_{11}^2\left[d_{22}^2 d_{33}^0 + d_{22}^0 d_{33}^2 - 2d_{23}^2 d_{23}^0 + d_{23}^1 d_{23}^1\right]$$
$$+ d_{12}^1\left[2d_{23}^3 d_{13}^0 - 2d_{23}^2 d_{13}^1 + 2d_{23}^1 d_{13}^2 + d_{12}^1 d_{33}^2\right]$$
$$+ d_{12}^0\left[-2d_{23}^3 d_{13}^1 + 2d_{23}^2 d_{13}^2 - d_{12}^0 d_{33}^4\right]$$
$$+ d_{22}^0\left[-d_{13}^2 d_{13}^2\right] + d_{22}^2\left[-2d_{13}^2 d_{13}^0 + d_{13}^1 d_{13}^1\right]\Big\}$$

$$g_2 = \frac{1}{\Delta}\Big\{ d_{11}^0\left[d_{22}^2 d_{33}^0 + d_{22}^0 d_{33}^2 - 2d_{23}^2 d_{23}^0 + d_{23}^1 d_{23}^1\right]$$
$$+ d_{11}^2\left[d_{22}^0 d_{33}^0 + d_{23}^0 d_{23}^0\right]$$
$$+ d_{12}^0\left[2d_{23}^2 d_{13}^0 - 2d_{23}^1 d_{13}^1 + 2d_{23}^0 d_{13}^2 - d_{12}^0 d_{23}^2\right]$$
$$+ d_{12}^1\left[2d_{23}^1 d_{13}^0 - 2d_{23}^0 d_{13}^1 + d_{12}^1 d_{33}^0\right]$$
$$+ d_{22}^0\left[-2d_{13}^2 d_{13}^0 + d_{13}^1 d_{13}^1\right] + d_{22}^2\left[-d_{13}^0 d_{13}^0\right]\Big\}$$

$$g_0 = \frac{1}{\Delta}\Big\{ d_{11}^0\left[d_{22}^0 d_{33}^0 - d_{23}^0 d_{23}^0\right]$$
$$+ d_{12}^0\left[2d_{23}^0 d_{13}^0 - d_{12}^0 d_{33}^0\right]$$
$$+ d_{22}\left[-d_{13}^0 d_{13}^0\right]\Big\}$$

and

$$\Delta = d_{11}^2 d_{22}^2 d_{33}^4 + d_{11}^2 d_{23}^3 d_{23}^3 .$$

All of the above are constants.

Equation (24) has eight roots, λ_{nj} $(j=1,2,\ldots,8)$.* The roots λ_{nj} may be real or complex. When they are complex, they occur in groups of four: $\lambda_n = \pm\chi_n \pm \mu_n i$

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* The method of solving equation (24) for λ_n is given in Reference (8).

while real roots come in pairs of opposite sign. Since the solution of real roots can be deduced from that of complex roots simply by letting μ_n equal to zero, a general solution of a group of complex roots of λ_n will be discussed in what follows.

Let a group of roots be

$$\lambda_{n1} = \chi_{n1} + i\,\mu_{n1} \qquad \lambda_{n3} = -\chi_{n1} + i\,\mu_{n1}$$
$$\lambda_{n2} = \chi_{n1} - i\,\mu_{n1} \qquad \lambda_{n4} = -\chi_{n1} - i\,\mu_{n1}.$$

On substitution of each one of the above roots into either two of equations (22), one may solve for A_{nj} and B_{nj} in terms of C_{nj} ($j=1,2,3$ and 4), such as

$$A_{nj} = \alpha_{nj} C_{nj} \qquad \text{and} \qquad B_{nj} = \beta_{nj} C_{nj}. \tag{25}$$

Now, one has the solution of (15) of this group of roots of λ_n as follows:

$$u = \sum_{n=1}^{\infty} [\alpha_{n1} C_{n1} y^{\lambda_{n1}-1} + \alpha_{n2} C_{n2} y^{\lambda_{n2}-1} + \alpha_{n3} C_{n3} y^{\lambda_{n3}-1} + \alpha_{n4} C_{n4} y^{\lambda_{n4}-1}] \cos\frac{n\pi\theta}{\theta_1}$$
$$V = \sum_{n=1}^{\infty} [\beta_{n1} C_{n1} y^{\lambda_{n1}-1} + \beta_{n2} C_{n2} y^{\lambda_{n2}-1} + \beta_{n3} C_{n3} y^{\lambda_{n3}-1} + \beta_{n4} C_{n4} y^{\lambda_{n4}-1}] \sin\frac{n\pi\theta}{\theta_1}$$
$$W = \sum_{n=1}^{\infty} [C_{n1} y^{\lambda_{n1}-1} + C_{n2} y^{\lambda_{n2}-1} + C_{n3} y^{\lambda_{n3}-1} + C_{n4} y^{\lambda_{n4}-1}] \sin\frac{n\pi\theta}{\theta_1}.$$

In the above solution, α_{nj}, β_{nj} and the undetermined constants C_{nj} are all complex numbers.

Because λ_{n1} is conjugate to λ_{n2}, and λ_{n3} is conjugate to λ_{n4}, one will expect that α_{n1} is conjugate to α_{n2} and α_{n3} conjugate to α_{n4}, i.e.

$$\alpha_{n1} = \bar{\alpha}_{n1} + i\,\bar{\alpha}_{n2} \qquad \alpha_{n3} = \bar{\alpha}_{n3} + i\,\bar{\alpha}_{n4}$$
$$\alpha_{n2} = \bar{\alpha}_{n1} - i\,\bar{\alpha}_{n2} \qquad \alpha_{n4} = \bar{\alpha}_{n3} - i\,\bar{\alpha}_{n4}.$$

And so for

$$\beta_{n1} = \bar{\beta}_{n1} + i\,\bar{\beta}_{n2} \qquad \beta_{n3} = \bar{\beta}_{n3} + i\,\bar{\beta}_{n4}$$
$$\beta_{n2} = \bar{\beta}_{n1} - i\,\bar{\beta}_{n2} \qquad \beta_{n4} = \bar{\beta}_{n3} - i\,\bar{\beta}_{n4}$$

where $\bar{\alpha}_{nj}$ and $\bar{\beta}_{nj}$ are resultant real number computed from either two of (22)*.

* Details see in Appendix I.

In order to express the solution in real form, a set of real numbers $\bar{C}_{nj}$ is also introduced such that

$$C_{n1} = \frac{1}{2}(\bar{C}_{n1} - i\bar{C}_{n2}) \qquad C_{n3} = \frac{1}{2}(\bar{C}_{n3} - i\bar{C}_{n4})$$
$$C_{n2} = \frac{1}{2}(\bar{C}_{n1} + i\bar{C}_{n2}) \qquad C_{n4} = \frac{1}{2}(\bar{C}_{n3} + i\bar{C}_{n4})$$

then α_{nj}, β_{nj}, C_{nj} and λ_{nj} may be replaced by their respective real constants $\bar{\alpha}_{nj}$, $\bar{\beta}_{nj}$, $\bar{C}_{nj}$, χ_n and μ_n. Using the identity

$$y^{i\mu} = e^{i\mu \ln y}$$

and those identities between the exponential functions and hyperbolic functions, one brings the above solution into real form.

$$\begin{aligned}
U = y^{-1}\sum_{n=1}^{\infty}\{ & y^{\chi_{n1}}[(\bar{\alpha}_{n1}\bar{C}_{n1} + \bar{\alpha}_{n2}\bar{C}_{n2})\cos(\mu_{n1}\ln y) + (\bar{\alpha}_{n1}\bar{C}_{n2} - \bar{\alpha}_{n2}\bar{C}_{n1})\sin(\mu_{n1}\ln y)] \\
& + y^{-\chi_{n1}}[(\bar{\alpha}_{n3}\bar{C}_{n3} + \bar{\alpha}_{n4}\bar{C}_{n4})\cos(\mu_{n1}\ln y) + (\bar{\alpha}_{n3}\bar{C}_{n4} - \bar{\alpha}_{n4}\bar{C}_{n3})\sin(\mu_{n1}\ln y)]\}\cos\frac{n\pi\theta}{\theta_1} \\
V = y^{-1}\sum_{n=1}^{\infty} & y^{\chi_{n1}}[(\bar{\beta}_{n1}\bar{C}_{n1} + \bar{\beta}_{n2}\bar{C}_{n2})\cos(\mu_{n1}\ln y) + (\bar{\beta}_{n1}\bar{C}_{n2} - \bar{\beta}_{n2}\bar{C}_{n1})\sin(\mu_{n1}\ln y)] \\
& + y^{-\chi_{n1}}[(\bar{\beta}_{n3}\bar{C}_{n3} + \bar{\beta}_{n4}\bar{C}_{n4})\cos(\mu_{n1}\ln y) + (\bar{\beta}_{n3}\bar{C}_{n4} - \bar{\beta}_{n4}\bar{C}_{n3})\sin(\mu_{n1}\ln y)]\}\sin\frac{n\pi\theta}{\theta_1} \\
W = y^{-1}\sum_{n=1}^{\infty}\{ & y^{\chi_{n1}}[(\bar{C}_{n1})\cos(\mu_{n1}\ln y) + (\bar{C}_{n2})\sin(\mu_{n1}\ln y)] \\
& + y^{-\chi_{n1}}[(\bar{C}_{n3})\cos(\mu_{n1}\ln y) + (\bar{C}_{n4})\sin(\mu_{n1}\ln y)]\}\sin\frac{n\pi\theta}{\theta_1} .
\end{aligned}$$

When this solution is introduced into equations (1), (3d) and (3f), it is found that all of the stress resultants of the homogeneous part may be brought into the form

$$F_m = \rho_m y^{Q_m} \sum_{n=1} F_{nm} \begin{matrix} \sin\frac{n\pi\theta}{\theta_1} \\ \cos\frac{n\pi\theta}{\theta_1} \end{matrix} , \tag{26a}$$

where

$$F_{nm} = y^{\chi_{n1}}[(A_{nm}\bar{C}_{n1} + B_{nm}\bar{C}_{n2})\cos(\mu_{n1}\ln y) + (A_{nm}\bar{C}_{n2} - B_{nm}\bar{C}_{n1})\sin(\mu_{n1}\ln y) + y^{-\chi_{n1}}[(G_{nm}\bar{C}_{n3} + H_{nm}\bar{C}_{n4})\cos(\mu_{n1}\ln y) + (G_{nm}\bar{C}_{n4} - H_{nm}\bar{C}_{n3})\sin(\mu_{n1}\ln y)] \tag{26b}$$

and m is a number to identify the stresses. The values f_m, Q_m and expressions A_{nm} and B_{nm} are given in Table I. The expressions G_{nm} and H_{nm} may be obtained by replacing $\bar{\alpha}_{n1}$, $\bar{\alpha}_{n2}$, $\bar{\beta}_{n1}$, β_{n2} and χ_{n1} in A_{nm} and B_{nm} by $\bar{\alpha}_{n3}$, α_{n4}, β_{n3}, β_{n4} and $-\chi_{n1}$ respectively, while μ_{n1} remains unaltered. When there is another set of complex roots of λ_n, say

$$\lambda_{n5} = \chi_{n2} + i\mu_{n2} \qquad \lambda_{n7} = -\chi_{n2} + i\mu_{n2}$$
$$\lambda_{n6} = \chi_{n2} - i\mu_{n2} \qquad \lambda_{n8} = -\chi_{n2} - i\mu_{n2}$$

the displacement components and stresses will be in the form

$$f_m = f_m y Q_m \sum_{n=1}^{\infty} f_{nm} \begin{matrix} \sin \frac{n\pi\theta}{\theta_1} \\ \cos \frac{n\pi\theta}{\theta_1} \end{matrix} \tag{27a}$$

where

$$f_{nm} = y^{\chi_{n2}}[(a_{nm}\bar{C}_{n5} + b_{nm}\bar{C}_{n6})\cos(\mu_{n2}\ln y) + (a_{nm}\bar{C}_{n6} - b_{nm}\bar{C}_{n5})\sin(\mu_{n2}\ln y) + y^{-\chi_{n2}}[(g_{nm}\bar{C}_{n7} + h_{nm}\bar{C}_{n8})\cos(\mu_{n2}\ln y) + (g_{nm}\bar{C}_{n8} - h_{nm}\bar{C}_{n7})\sin(\mu_{n2}\ln y) \tag{27b}$$

in which $\bar{C}_{ni}$ (i=5,6,7 and 8) are four arbitrary constants. To obtain the expressions a_{nm}, b_{nm}, g_{nm} and h_{nm}, $\bar{\alpha}_{ni}$ and $\bar{\beta}_{ni}$ are computed first by replacing χ_{n1} and μ_{n1} by χ_{n2} and μ_{n2} in the computations of $\bar{\alpha}_{nj}$ and $\bar{\beta}_{nj}$ then replacing $\bar{\alpha}_{nj}$, $\bar{\beta}_{nj}$, χ_{n2} and μ_{n2} in the expressions of A_{nm}, B_{nm}, G_{nm} and H_{nm} by $\bar{\alpha}_{ni}$, $\bar{\beta}_{ni}$, χ_{n2} and μ_{n2}.

When the roots of λ_n are real, the formulas may be obtained by putting $\chi_{n1}=\lambda_{n1}$, $\mu_{n1}=0$, $\bar{\alpha}_{n1}=\alpha_{n1}$, $\bar{\beta}_{n1}=\beta_{n1}$ and $\bar{\alpha}_{n2}=\bar{\beta}_{n2}=0$. Then $B_{nm}=0$, A_{nm} are the coefficients to be used in (26). By the similar replacement one can obtain the coefficients G_{nm}, a_{nm} and g_{nm}. Repeating this procedure for the other pairs of roots of λ_n, one gets the homogeneous solution.

A particular solution is obtained in what follows when the lateral normal load distribution in the s-direction, P(s) in (16), is assumed in the form

$$P(s)=s^r$$

where r is a constant. However, for simplicity, r is assumed to be a real number in the following solution. Changing the variable s to y according to (17), one has

$$P(y)=\ell^r y^{2r}. \tag{28}$$

On substitution of the above into equation (19) and observing (19c), one finds

$$\lambda_n - 1 = 2(r+2)$$

or

$$\lambda_n = 2r+5 \equiv \bar{\lambda} \tag{29}$$

and

$$\begin{aligned} d_{11}\bar{A}_n + d_{12}\bar{B}_n + d_{13}\bar{C}_n &= 0 \\ d_{21}\bar{A}_n + d_{22}\bar{B}_n + d_{23}\bar{C}_n &= 0 \\ d_{31}\bar{A}_n + d_{32}\bar{B}_n + d_{33}\bar{C}_n &= J a_n \end{aligned} \tag{30}$$

where $J=\frac{L^{r+2}}{D}$, a given constant. The hyphens are placed above the constants A_n, B_n and C_n to distinguish them from those in the homogeneous solution. The λ_n in expressions (20) is now a known number $\bar{\lambda}$, hence all the coefficients in (3) can be computed by following (20). From (30), one can solve for A_n, B_n and C_n. Let

$$\Omega \equiv \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

Then

$$\bar{C}_n = \frac{1}{\Omega} J\, a_n [d_{11}d_{22} - d_{12}d_{21}] \tag{31}$$

Expressing $\bar{A}_n$ and $\bar{B}_n$ in terms of $\bar{C}_n$, one has

$$\bar{A}_n = \phi_n \bar{C}_n \quad \text{and} \quad \bar{B}_n = \psi_n \bar{C}_n \tag{32}$$

where

$$\phi_n = \frac{d_{12}d_{23} - d_{13}d_{22}}{d_{11}d_{22} - d_{12}d_{21}}$$

$$\psi_n = \frac{d_{13}d_{21} - d_{11}d_{23}}{d_{11}d_{22} - d_{12}d_{21}}$$

Thus, the particular solutions of the displacements are in the following form:

$$\begin{aligned} U_p &= y^{\bar{\lambda}-1} \sum_{n=1}^{\infty} \phi_n \bar{C}_n \cos \frac{n\pi\theta}{\theta_1} \\ V_p &= y^{\bar{\lambda}-1} \sum_{n=1}^{\infty} \psi_n \bar{C}_n \sin \frac{n\pi\theta}{\theta_1} \\ W_p &= y^{\bar{\lambda}-1} \sum \bar{C}_n \sin \frac{n\pi\theta}{\theta_1} \end{aligned} \tag{33}$$

Substituting these solutions into the elastic law (1), the particular-solution part of the stresses may be given in the following form:

$$I_m = f_m y^{a_m} \sum_{n=1}^{\infty} I_{nm} \begin{matrix} \sin \frac{n\pi\theta}{\theta_1} \\ \cos \frac{n\pi\theta}{\theta_1} \end{matrix} \tag{34a}$$

where

$$I_m = S_{nm} \bar{C}_n y^{\bar{\lambda}} \tag{34b}$$

f_m and Q_m are again given in Table I. S_{nm} is obtained by putting $\chi_n = \bar{\lambda}$, $\mu_1 = 0$, $\alpha_{n1} = \phi_n$, $\beta_{n1} = \psi_n$ and $\alpha_{n2} = \beta_{n2} = 0$ of A_{nm} in the table.

Combine the solutions given by (26), (27) and (34), the complete solution is obtained. Let this solution be expressed in the form

$$\bar{X}_m = \tau_m y^{a_m} \sum_{n=1}^{\infty} X_{nm} \begin{matrix} \sin \frac{n\pi\theta}{\theta_1} \\ \cos \frac{n\pi\theta}{\theta_1} \end{matrix} \tag{35a}$$

where

$$X_{nm} = F_{nm} + f_{nm} + I_{nm} \tag{35b}$$

According to the boundary conditions given by (8), (9) and (11), one requires that

$$\begin{aligned} \bar{X}_{n1} = \bar{X}_{n2} = \bar{X}_{n3} = \bar{X}_{n4} &= 0 \qquad \text{for} \quad y = \sqrt{\frac{L_1}{L}} \\ \bar{X}_{n5} = \bar{X}_{n9} = \bar{X}_{n15} = \bar{X}_{n16} &= 0 \qquad \text{for} \quad y = 1 \end{aligned} \tag{36}$$

This set of eight simultaneous equations enables one to determine the eight arbitrary constants $\bar{C}_{nk}$ (k=1,2,....8) involved in F_{nm} and f_{nm} for each n. With these constants determined, one obtains all the desired stresses from (35).

III. DISCUSSION AND CONCLUSION

The bar required to be attached to each of the straight edges would offer a certain degree of stiffness in the transverse direction. Nevertheless, it cannot be considered as a support. This discrepancy makes the solution to be an approximate one; otherwise, it is exact. The accuracy of the solution increases with the increase of the transverse stiffness of the bar. The effect of this approximation would be secondary in stresses as it has been seen from the order examination. However, due to the relatively weak stiffenss of the edges, the effect on the deflection would be considerable. An up-bound of the deflection could be given, in case it is of interest, by considering the bar as a cantilever beam subjected to the transverse shearing force.

For very thin shells, the constant k defined in (6) is very small. Hence, some terms of its product might be omitted. However, for generality and based on the consideration that the present solution can only be worked out through the use of a high speed computer, the retention of these terms would not cause too much trouble. Therefore all of the terms are remained.

Before a conclusion is made, a numerical example should be made. This will be done in an extention of the work.

APPENDIX I

DETAILED PROCEDURES FOR APPLYING THE SOLUTION TO A STIFFENED SEGMENT OF A SHELL

The obtained solution of the unstiffened segment may be applied to a stiffened segment simply by replacing the thickness t, rigidity constants D and K defined in (2) and the constant k in (6) by their equivalences, if the stiffeners are closely spaced and arranged in a same way in both s and θ -directions. The equivalent constants may be computed in the following way. Let the geometrical dimensions and material constants of a given stiffened segment

$$\alpha, \theta_1, L, L_1, \nu \quad \text{and } E \tag{A1}$$

be given and let a_1 be the cross-sectional area of a stiffener and a_2 be the area of the wall of the segment between two neighboring stiffeners which are spaced with a distance b. Then the equivalent thickness t is given by

$$t_s = \frac{a_1 + a_2}{b} \tag{A2}$$

The equivalent moment of inertia per unit width is $\frac{I}{b}$, in which I is the moment of inertia of the composite section of a_1 and a_2. Put

$$\frac{I}{b} = c\,\frac{t_s^3}{12} \tag{A3}$$

from which one can compute the constant c. The equivalent rigidity constants D and K may be obtained by following (2) and (A3):

$$D = \frac{E t_s}{1-\nu} \quad , \quad K_s = \frac{c\,E\,t_s^3}{12(1-\nu^2)} \tag{A4}$$

Assuming

$$t_s = \delta \cdot s \tag{A5}$$

one has

$$\frac{K_s}{D_s} = k_s S^2$$

where

$$k_s = C\frac{\delta^2}{12} \tag{A6}$$

Thus, by replacing the constants D, K and k appearing in the solution by D_s, K_K and k_s respectively, one can compute the constants d_{ij}^{K} defined in (21) and in turn the coefficients g_i given in (24) for each n (n=1,2,....,see late).

To solve equation (24) for λ_n, following the method given in reference (9), let

$$\lambda_n^2 = X$$

$$g_6 = B,\ g_4 = C,\ g_2 = D\ ,\ g_0 = E \tag{A7}$$

Then equation (24) reads

$$X^4 + BX^3 + CX^2 + DX + E = 0 \tag{A8}$$

which can be separated into two equations

$$X^2 + \left(\frac{B}{2} + \sqrt{\frac{B^2}{4} - (C - z)}\right)X + \left(\frac{z}{2} + \sqrt{\frac{z^2}{4} - E}\right) = 0$$

and

$$X^2 + \left(\frac{B}{2} - \sqrt{\frac{B^2}{4} - (C - z)}\right)X + \left(\frac{z}{2} - \sqrt{\frac{z^2}{4} - E}\right) = 0 \tag{A9}$$

where z satisfies

$$z = C - \frac{R}{z^2 + A}$$

and

$$R = BCD - D^2 - B^2E$$

$$A = BD - 4E$$

The approximate solution of z may be obtained by the rule

$$z_{n+1} = C - \frac{R}{z_n^2 + A}$$

where z_n is the nth approximation considering $z_0 = 0$. In case the convergence of the approximation is slow, Newton s method may be applied. If $z = p$ be any approximate solution, a better approximation is

$$Z = P - \frac{(P-C)(P^2+A)+R}{(P^2+A)+2P(P-C)}$$

Therefore, from equations (A9) and (A7), one obtains eight roots of λ_n, say λ_{nj} ($j=1,2,\ldots,8$). The next step is to compute α_{nj} and β_{nj} as defined in (25). They may be computed from any two of equations (22). Let the first two be used which, using the coefficients given in (20), become

$$[d_{11}^2 \lambda_{nj}^2 + d_{11}^0] A_n + [d_{12}' \lambda_{nj} + d_{12}^0] B_n + [d_{13}^2 \lambda_{nj}^2 + d_{13}' \lambda_{nj} + d_{13}^0] C_n = 0 \tag{A10}$$

$$[-d_{12}' \lambda_{nj} + d_{12}^0] A_n + [d_{22}^2 \lambda_{nj}^2 + d_{22}^0] B_n + [d_{23}^3 \lambda_{nj}^3 + d_{23}^2 \lambda_{nj}^2 + d_{23}' \lambda_{nj} + d_{23}^0] C_n = 0$$

In general

$$\lambda_{nj} = x_{nj} + i\mu_{nj}$$

$$\lambda_{nj}^2 = (x_{nj}^2 - \mu_{nj}^2) + 2i x_{nj}\mu_{nj}$$

$$\lambda_{nj}^3 = (x_{nj}^3 - 3x_{nj}\mu_{nj}^2) + i(3x_{nj}^2\mu_{nj} - \mu_{nj}^3)$$

NOTE: $x_{nj} = x_{n1}$ for $j=1,2$; $x_{nj} = -x_{n1}$ for $j=3,4$, etc. *see late.*

Hence, (A10) may be written in the following form

$$[a_1 + i\bar{a}_1] A_n + [b_1 + i\bar{b}_1] B_n = -[s_1 + i\bar{s}_1] C_n \tag{A11}$$

$$[a_2 + i\bar{a}_2] A_n + [b_2 + i\bar{b}_2] B_n = -[s_2 + i\bar{s}_2] C_n$$

where

$$a_1 = d_{11}^2 (x_{nj}^2 - \mu_{nj}) + d_{11}^0$$

$$\bar{a}_1 = 2d_{11}^2 x_{nj}\mu_{nj}$$

$$b_1 = d_{12}' x_{nj} + d_{12}^0$$

$$\bar{b}_1 = d_{12}' \mu_{nj}$$

$$s_1 = d_{13}^2 (x_{nj}^2 - \mu_{nj}^2) + d_{13}' x_{nj} + d_{13}^0$$

$$s_2 = [2d_{13}^2 x_{nj} + d_{13}'] \mu_{nj} \tag{A12}$$

$$a_2 = -d_{12}' \chi_{nj} + d_{12}^0$$
$$\bar{a}_2 = -d_{12}' \mu_{nj}$$
$$b_2 = d_{22}^2(\chi_{nj}^2 - \mu_{nj}^2) + d_{22}^0$$
$$\bar{b}_2 = 2d_{22}^2 \chi_{nj} \mu_{nj}$$
$$s_2 = d_{23}^3(\chi_{nj}^3 - 3\chi_{nj}\mu_{nj}^2) + d_{23}^2(\chi_{nj}^2 - \mu_{nj}^2) + d_{23}' \chi_{nj}' + d_{23}^0$$
$$\bar{s}_2 = [d_{23}^3(3\chi_{nj}^2 - \mu_{nj}^2) + 2d_{23}^2 \chi_{nj} + d_{23}']\mu_{nj}$$

Now, A_n and B_n are to be solved from (A11). Let

$$\Delta = \begin{vmatrix} a_1 + i\bar{a}_1 & b_1 + i\bar{b}_1 \\ a_2 + i\bar{a}_2 & b_2 + i\bar{b}_2 \end{vmatrix}$$
$$= [a_1 b_2 - \bar{a}_1\bar{b}_2 - a_2 b_1 + \bar{a}_2\bar{b}_1] + i[\bar{a}_1 b_2 + a_1\bar{b}_2 - a_2\bar{b}_1 - \bar{a}_2 b_1]$$
$$= \Delta_1 + i\bar{\Delta}_1$$

$$D = \begin{vmatrix} c_1 + i\bar{c}_1 & b_1 + i\bar{b}_1 \\ c_2 + i\bar{c}_2 & b_2 + i\bar{b}_2 \end{vmatrix}$$
$$= [c_1 b_2 - \bar{c}_1\bar{b}_2 - b_1 c_2 + \bar{b}_1\bar{c}_2] + i[c_1\bar{b}_2 + \bar{c}_1 b_2 - b_1\bar{c}_2 - \bar{b}_1 c_2]$$
$$= D_1 + i\bar{D}_1$$

$$E = \begin{vmatrix} a_1 + i\bar{a}_1 & c_1 + i\bar{c}_1 \\ a_2 + i\bar{a}_2 & c_2 + i\bar{c}_2 \end{vmatrix}$$
$$= [a_1 c_2 - \bar{a}_1\bar{c}_2 - a_2 c_1 + \bar{a}_2\bar{c}_1] + i[\bar{a}_1 c_2 + a_1\bar{c}_2 - a_2\bar{c}_1 - \bar{a}_2 c_1]$$
$$= E_1 + i\bar{E}_1$$

where Δ_1, $\bar{\Delta}_1$, D_1, $\bar{D}_1$, E and $\bar{E}_1$ are defined in their respective equations. Then, one has the roots of A_n and B_n as

$$A_{nj} = \alpha_{nj}\, C_n \qquad \text{and} \qquad B_{nj} = \beta_{nj}\, C_n$$

where

$$\alpha_{nj} = -\frac{D}{\Delta} = -\left[\frac{D_1\Delta_1 + \bar{D}_1\bar{\Delta}_1}{\Delta_1^2 + \bar{\Delta}_1^2}\right] - i\left[\frac{\bar{D}_1\Delta_1 - D_1\bar{\Delta}_1}{\Delta_1^2 + \bar{\Delta}_1^2}\right]$$

$$\beta_{nj} = -\frac{E}{\Delta} = -\left[\frac{E_1\Delta_1 + \bar{E}_1\bar{\Delta}_1}{\Delta_1^2 + \bar{\Delta}_1^2}\right] + i\left[\frac{\bar{E}_1\Delta_1 - E_1\bar{\Delta}_1}{\Delta_1^2 + \bar{\Delta}_1^2}\right] \qquad \text{(A13)}$$

It may be seen from (A12) to (A13) that if one changes the sign of μ_{nj} from positive to negative, only the sign of the imaginary parts of all the expressions will be changed, while the real parts will not be affected. But when one changes the sign of χ_{nj} not only the signs but the magnitudes also will be affected. Hence, if one assumes a group of λ_{nj} to be

$$\lambda_{n1} = \chi_{n1} + i\mu_{n1} \qquad \lambda_{n3} = -\chi_{n1} + i\mu_{n1}$$
$$\lambda_{n2} = \chi_{n1} - i\mu_{n1} \qquad \lambda_{n4} = -\chi_{n1} - i\mu_{n1}$$

one will have

$$\alpha_{n1} = \bar{\alpha}_{n1} + i\bar{\alpha}_{n2} \qquad \alpha_{n3} = \bar{\alpha}_{n3} + i\bar{\alpha}_{n4}$$
$$\alpha_{n2} = \bar{\alpha}_{n1} - i\bar{\alpha}_{n2} \qquad \alpha_{n4} = \bar{\alpha}_{n3} - i\alpha_{n4}$$

and

$$\beta_{n1} = \bar{\beta}_{n1} + i\bar{\beta}_{n2} \qquad \beta_{n3} = \bar{\beta}_{n3} + i\bar{\beta}_{n4}$$
$$\beta_{n2} = \bar{\beta}_{n1} - i\beta_{n2} \qquad \beta_{n4} = \beta_{n3} - i\beta_{n4}$$

where $\bar{\alpha}_{ni}$ and $\bar{\beta}_{ni}$ (i=1,2,3 and 4) are computed from (A13).

With $\bar{\alpha}_{n1}$, $\bar{\alpha}_{n2}$, $\bar{\beta}_{n1}$, $\bar{\beta}_{n2}$, χ_{n1} and μ_{n1} at hand, one is ready to go to Table I to compute A_{nm} and B_{nm} in (26). As has been stated before, by replacing $\bar{\alpha}_{n1}$, $\bar{\alpha}_{n2}$, $\bar{\beta}_{n1}$, $\bar{\beta}_{n2}$ and χ_{n1} by α_{n3}, α_{n4}, $\bar{\beta}_{n3}$, $\bar{\beta}_{n4}$ and $-\chi_n$ in A_{nm} and B_{nm}, one obtains G_{nm} and H_{nm} respectively. For the other set of complex roots of λ_n, or when the roots are real, these expressions may be obtained by following the procedures stated after equation (27). This completes the computation of the homogeneous solution.

Consider the case in which the lateral normal load is uniformly distributed in the s-direction and is a step function in θ-direction such that

$$p = p_1 \quad \text{for} \quad 0 \leq \theta < \frac{\theta_1}{2}$$
$$p = p_2 \quad \text{for} \quad \frac{\theta_1}{2} < \theta \leq \theta_1$$

Hence $\gamma = 0$ in (28). The a_n in (16) may be computed as coefficients of Fourier series. For the above load,

$$a_n = \frac{P_1 2}{n\pi}\left[\left(\frac{P_2}{P_1}\cos n\pi - 1\right) + \left(1 - \frac{P_2}{P_1}\right)\cos\frac{n\pi}{2}\right]$$

or

$$a_n = \frac{2P_1}{n\pi}\left[-\left(\frac{P_2}{P_1}+1\right)\right] \quad n = 1,3,5\cdots \qquad a_n = \frac{4P_1}{n\pi}\left[\frac{P_2}{P_1}-1\right] \quad n = 2,6,10\cdots$$

$$a_n = 0 \quad , \quad n = 4,8,12\cdots$$

The constants $\bar{\lambda}$ and J which appeared in (29) and (30), now are

$$\bar{\lambda} = 5 \qquad \text{and} \qquad J = \frac{1}{D_s} L^2$$

Hence, one may compute $\bar{C}_n$, ϕ_n and ψ_n as given by (31) and (32). Then letting $\chi_{n1} = \bar{\lambda}$, $\mu_{n1} = 0$, $\bar{\alpha}_{n1} = \phi_n$, $\bar{\beta}_{n1} = \psi_n$ and $\bar{\alpha}_{n2} = \bar{\beta}_{n2} = 0$ of A_{nm} in Table I, one obtains the expressions S_{nm} in (34). This completes the particular solution and one may write down the complete solution as follows:

$$\begin{aligned} X_m = \rho_m y^{Q_m} \sum_{n=1}^{\infty} \{ & y^{\chi_{n1}}[(A_{nm} C_{n1} + B_{nm}\bar{C}_{n2})\cos(\mu_{n1} \ln y) + (A_{nm}\bar{C}_{n2} - B_{nm}\bar{C}_{n1})\sin(\mu_{n1}\ln y)] \\ & + y^{-\chi_{n1}}[(G_{nm}\bar{C}_{n3} + H_{nm}\bar{C}_{n4})\cos(\mu_{n1}\ln y) + (G_{nm}\bar{C}_{n4} - H_{nm}\bar{C}_{n3})\sin(\mu_{n1}\ln y)] \\ & + y^{\chi_{n2}}[(a_{nm}\bar{C}_5 + b_{nm}\bar{C}_6)\cos(\mu_{n2}\ln y) + (a_{nm}\bar{C}_6 - b_{nm}\bar{C}_5)\sin(\mu_{n2}\ln y)] \\ & + y^{-\chi_{n2}}[(g_{nm}\bar{C}_7 + h_{nm}\bar{C}_8)\cos(\mu_{n2}\ln y) + (g_{nm}\bar{C}_8 - h_{nm}\bar{C}_7)\sin(\mu_{n2}\ln y)] \end{aligned}$$

$$+ y^{\bar{\lambda}} S_{nm} C_n \Big\} \begin{matrix} \sin \frac{n\pi\theta}{\theta_1} \\ \cos \frac{n\pi\theta}{\theta_1} \end{matrix} \tag{A 14}$$

The first set of four boundary conditions in (36) for $y = \xi$, where $\xi = \sqrt{\frac{L_1}{L}}$, if they are put in a complete form, reads

$$\begin{aligned}
&\xi^{\chi_{n1}-\bar{\lambda}}\Big\{[A_{ni}\cos(\mu_{n1}\ln\xi) - B_{ni}\sin(\mu_{n1}\ln\xi)]\,\bar{C}_{n1} \\
&\qquad + [B_{ni}\cos(\mu_{n1}\ln\xi) + A_{ni}\sin(\mu_{n1}\ln\xi)]\,\bar{C}_{n2}\Big\} \\
&+\xi^{-\chi_{n1}-\bar{\lambda}}\Big\{[G_{ni}\cos(\mu_{n1}\ln\xi) - H_{ni}\sin(\mu_{n1}\ln\xi)]\,\bar{C}_{n3} \\
&\qquad + [H_{ni}\cos(\mu_{n1}\ln\xi) + G_{ni}\sin(\mu_{n1}\ln\xi)]\,\bar{C}_{n4}\Big\} \\
&+\xi^{\chi_{n2}-\bar{\lambda}}\Big\{[a_{ni}\cos(\mu_{n2}\ln\xi) - b_{ni}\sin(\mu_{n2}\ln\xi)]\,\bar{C}_{n5} \\
&\qquad + [b_{ni}\cos(\mu_{n2}\ln\xi) + a_{ni}\sin(\mu_{n2}\ln\xi)]\,\bar{C}_{n6}\Big\} \\
&+\xi^{-\chi_{n2}-\bar{\lambda}}\Big\{[g_{ni}\cos(\mu_{n2}\ln\xi) - h_{ni}\sin(\mu_{n2}\ln\xi)]\,\bar{C}_{n7} \\
&\qquad + [h_{ni}\cos(\mu_{n2}\ln\xi) + g_{ni}\sin(\mu_{n2}\ln\xi)]\,\bar{C}_{n8} = -S_{ni}\bar{C}_n
\end{aligned}$$

for every n and where i = 1,2,3, and 4. The other set of four conditions, for y = 1, is

$$\begin{aligned}
&A_{nk}\bar{C}_{n1} + B_{nk}\bar{C}_{n2} + G_{nk}\bar{C}_{n3} + H_{nk}\bar{C}_{n4} \\
&+ a_{nk}\bar{C}_{n5} + b_{nk}\bar{C}_{n6} + g_{nk}\bar{C}_{n7} + h_{nk}\bar{C}_{n8} = -S_{nk}\bar{C}_n
\end{aligned}$$

where k = 5, 9, 15 and 16 and also for every n. The eight constants $\bar{C}_{nj}$ are readily determined from these eight simultaneous equations. Substitutint these $\bar{C}_{nj}$ back into (A14), one obtains all the stresses. However, it must be noted that for the present stiffened segment, the coefficients f_m in Table I, for m from 9 to 17 inclusive have to be modified by multiplying them by the constant c given in (A3).

It is seen from Table I that the maximum normal stress and moment occur at $s = L_1$ and $\theta = \frac{\theta_1}{2}$. Hence, for design purpose, the resultant stresses and moments of the stiffener at that point may be obtained by multiplying the stress and moments at that point by the spacing between the two neighboring stiffeners, b.

To each of the two straight edges, a bar is required to be attached. This bar is subjected to the axial shearing force $N_{\theta s}$ and transverse shearing force S_θ. In order to fulfill the assumption that the straight edges of the segment are free to rotate and to move in the tangential direction, the connection between the bar and the edge is required to be designed accordingly.

APPENDIX II. NOTATION

The following symbols have been adopted for use in this report:

a_n = Fourier coefficients;

a_{nm}, b_{nm} = coefficients defined in equation (27b);

A_n, B_n, C_n = coefficients of solutions of u, v and w in equations (15);

$\overline{A}_n$, $\overline{B}_n$, $\overline{C}_n$ = coefficients of the particular solutions of u, v and w;

A_{nm}, B_{nm} = coefficients defined in equation (26b) and given in Table I;

A_{nj}, B_{nj}, C_{nj} = coefficients of solutions of u, v and w associated with the root λ_{nj};

d_{ij} = coefficients defined in equations (20);

d_{ij}^{k} = coefficients defined in equations (21);

D = constant defined in equation (2);

D_s = constant defined in equation (A4);

E = Young's modulus of elasticity;

g_n = Coefficients defined in equation (24);

F_m, f_m = general forms of homogeneous solutions associated with λ_{n1} to λ_{n4} and λ_{n5} to λ_{n8} respectively;

F_{nm}, f_{nm} = functions defined in equations (26b) and (27b) respectively;

G_{nm}, H_{nm} = coefficients defined in equation (26b);

g_{nm}, h_{nm} = coefficients defined in equation (27b);

I_{nm} = coefficient defined in equation (34);

J = constant defined in equations (30);

k = constant defined in equation (6);

k_s = constant defined in equation (A5);

K = bending rigidity defined in equations (2);

K_s = bending rigidity of stiffened shell defined in equation (A4);

L, L_1 = distances measured from the apex along the conical surface to the fixed end and free end of the segment of the cone respectively;

M_s M_θ = normal moments per unit length in planes perpendicular to s and θ directions

$M_{s\theta}$, $M_{\theta s}$ = Twisting moments per unit length in planes perpendicular to s and θ directions respectively;

N_s, N_θ = Normal Forces per unit length in planes perpendicular to s and θ directions respectively;

$N_{s\theta}$, $N_{\theta s}$ = tangential shearing forces per unit in planes perpendicular to s and θ directions respectively;

P_s, P_θ , P_r = surface loads per unit area in the directions of s, θ and the normal to the middle surface respectively;

Q_s, Q_θ = transverse shearing forces in planes perpendicular to s and θ directions respectively per unit length;

Q_m = coefficients defined in equation (26) and given in Table I;

s = distance measured from the apex along the conical surface;

S_{nm} = coefficients defined in equation (34b);

S_s, S_θ = resultant transverse shearing forces due to Q_s, Q_θ $M_{s\theta}$ and $M_{\theta s}$;

t = thickness of shell;

t_s = equivalent thickness of stiffened shell defined in equation (A2);

T_s, T_θ = resultant tangential shearing forces due to $N_{s\theta}$, $N_{\theta s}$ $M_{s\theta}$ and $M_{\theta s}$;

u,v,w = components of displacement in the directions of s, θ and the normal to the middle surface directions respectively;

u_p,v_p,w_p = particular solutions of u, v and w;

$\ddot{X}_m$ = general solutions defined in equation (35a);

$y = \sqrt{\frac{s}{L}}$

α = angle between the conical surface and a plane perpendicular to the axis of the cone;

α_{nj}, β_{nj} = coefficients defined in equations (25);

$\bar{\alpha}_{nj}, \bar{\beta}_{nj}$ = real numbers of α_{nj} and β_{nj} ;

γ = constant given in equation (28);

δ = constant defined in equation (5);

θ = angle between two meridians;

θ_1 = angle between the two extreme meridians of the shell segment;

λ_n = undetermined constant given in equation (18);

λ_{nj} = jth root of λ_n

$\bar{\lambda}$ = known constant given by equation (29);

$\mu_n, \mu_{n1}, \mu_{n2}$ = imaginary parts of λ_n and λ_{nj} ;

$\varkappa_n, \varkappa_{n1}, \varkappa_{n2}$ = real parts of λ_n and λ_{nj} ;

ν = Poisson's ratio;

ρ_m = coefficient defined in equation (26a) and given in Table 1.

Differentiation Notation: Differentiation with respect to s and θ coordiantes are indicated by dot "." and prime "," respectively.

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